Boundary feedback controller over a bluff body for prescribed drag and lift coefficients

Evrad Marie Diokel Ngom, Abdou Sène, Daniel Le Roux

Boundary feedback controller over a bluff body for prescribed drag and lift coefficients

Evrad M. D. Ngom[a,c,*], Abdou Sène[b], Daniel Y. Le Roux[c]

[a] *UFR de Sciences Appliquées et Technologie, Université Gaston Berger, B.P. 234 Saint-Louis, Sénégal*
[b] *CEA-MITIC, Université Gaston Berger B.P. 234 Saint-Louis, Sénégal*
[c] *Université de Lyon, CNRS, Université Lyon 1, Institut Camille Jordan, 43, blvd du 11 novembre 1918, 69622 Villeurbanne Cedex, France*

Abstract

This paper presents an improved boundary feedback controller for the two and three-dimensional Navier-Stokes equations, in a bounded domain Ω, *for prescribed drag and lift coefficients.* In order to determine the feedback control law, we consider an extended system coupling the equations governing the Navier-Stokes problem with an equation satisfied by the control on the bluff body, which is a part of the domain boundary. By using the Faedo-Galerkin method and a priori estimation techniques, a stabilizing boundary control is built. This control law ensures the stability of the controlled discrete system. A compactness result then allows us to pass to the limit in the non linear system satisfied by the approximated solutions.

Keywords: Navier-Stokes system, boundary feedback stabilization, bluff body, drag and lift coefficients

1. Introduction

Flow over a bluff body is a common occurrence associated with fluid flowing over an obstacle or with the movement of a natural or artificial body. Evident examples are the flows past an airplane, a submarine, and wind blowing past a bridge or a high-rise building. This paper presents an improved boundary feedback control for the two and three-dimensional Navier-Stokes equations around a bluff body. Let Ω be a bounded and connected domain in $\mathbb{R}^d$ $(d = 2, 3)$, with a boundary Γ of class C^2, and composed of two connected components Γ_b and Γ_c such that $\Gamma = \Gamma_b \cup \Gamma_c$. Such a boundary decomposition is schematized in Figure 1. In particular, the boundary Γ_c represents the contour of the bluff body, and it is the part of Γ where a Dirichlet boundary control in feedback form has to be determined.

For $\mathbf{e}_i = (\delta_{1i}, \delta_{2i}, \delta_{3i}),\ i = 1, \cdots, d$, with δ_{ij} the Kronecker symbol, Γ_c is chosen such that

$$\int_{\Gamma_c} \mathbf{e}_i \cdot \mathbf{n}\, d\zeta = 0 \tag{1.1}$$

where $\mathbf{n}$ denotes the unit outer normal vector to Γ.

*Corresponding author

Email addresses: dioxleroi@hotmail.com (Evrad M. D. Ngom), abdou.sene@ugb.edu.sn (Abdou Sène), dleroux@math.univ-lyon1.fr (Daniel Y. Le Roux)

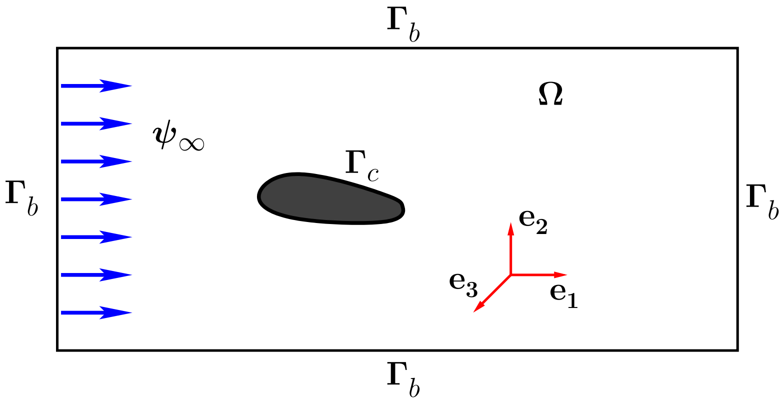

Figure 1: Description of the domain Ω and of the two connected components Γ_b and Γ_c.

For example, condition (1.1) holds when Γ_c is a sphere with center $(0,0,0)$ and radius r. Indeed, in that case, Γ_c is the locus of all points $\mathbf{X} = (x,y,z)^t$ such that $f(\mathbf{X}) = \|\mathbf{X}\|^2 - r^2 = 0$, which leads to

$$\mathbf{n} = -\frac{\nabla f(\mathbf{X})}{\|\nabla f(\mathbf{X})\|} = -\frac{\mathbf{X}}{\|\mathbf{X}\|},$$

and hence, (1.1) is obtained. Condition (1.1) also holds in the case where $f(x,y,z)$ is the contour of a circular cylinder. More generally, when $f(x,y,z)$ represents the boundary Γ_c, condition (1.1) is satisfied if $\nabla f(x,y,z)$ is odd with respect to each variable x,y,z, supplemented with specific symmetries for $f(x,y,z)$. In fact, condition (1.1) is satisfied for a number of contours $f(\mathbf{X})$, including, for example, the equation of a symmetrical 4-digit NACA airfoil [28, page 7].

Let $T > 0$ be a fixed real number, $Q = [0,T[\times\Omega$, $\Sigma_b = [0,T[\times\Gamma_b$, $\Sigma_c = [0,T[\times\Gamma_c$ and $\mathbf{V}^{1/2}(\widetilde{\Gamma})$, $\widetilde{\Gamma} \subset \Gamma$, is defined as the space of trace functions whose extension by zero over Γ, belongs to $\mathbf{H}^{1/2}(\Gamma)$. We consider the perturbed trajectory $(\mathbf{u},\pi)$, solution of the non-stationary Navier-Stokes model

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\,\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\pi = \mathbf{f} & \text{in } Q, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u} = \boldsymbol{\psi}_\infty(\mathbf{x}) & \text{on } \Sigma_b, \\ \mathbf{u} = \mathbf{v}_c(t,\mathbf{x}) & \text{on } \Sigma_c, \\ \mathbf{u}(t=0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\mathbf{u}$ and π are the velocity field and the pressure, respectively, ν is the kinematic viscosity, and $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ represents body forces acting on the fluid. Further, $\mathbf{u}_0(\mathbf{x})$ is the initial condition, and $\mathbf{v}_c(t,\mathbf{x})$ represents the control input on Σ_c, while the specified Dirichlet boundary condition $\boldsymbol{\psi}_\infty$ is such that

$$\boldsymbol{\psi}_\infty \in \mathbf{V}^{1/2}(\Gamma_b) \quad \text{and} \quad \int_{\Gamma_b} \boldsymbol{\psi}_\infty\cdot\mathbf{n}\,d\zeta = 0. \tag{1.3}$$

The different regimes of the flow are given by the values of the Reynolds number $\mathcal{R}_e = \frac{\overline{\psi}_\infty D}{\nu}$, with

D and $\overline{\boldsymbol{\psi}}_\infty$ being the characteristic dimension (e.g., the size of Γ_c) and the characteristic velocity, respectively.

For low Reynolds numbers, the fluid over the body is highly viscous, and the force exerted on the body is mainly attributed to skin friction. However, when the Reynolds number $\mathcal{R}_e$ exceeds a certain critical value, small perturbations destabilize the solution of the system (1.2) and yield a periodic solution $(\mathbf{u}, \pi)$ represented by the well-known von Kármán vortex street. In fluid dynamics, a von Kármán vortex street is a repeating pattern of swirling vortices caused by the unsteady separation of flow of a fluid around blunt bodies. This vortex shedding is responsible for such phenomena as the "singing" of suspended telephone or power lines, and the vibration of a car antenna at certain speeds which may lead to structural failure or reduction in performance. Further, vortex shedding occurs over a wide range of Reynolds numbers, causing significant increases in the mean drag and lift fluctuations. Therefore, the effective control of vortex shedding is important in engineering applications.

Recall that in fluid dynamics, the drag coefficient, denoted by C_x, is a dimensionless quantity that is used to quantify the drag or resistance of an object in a fluid environment, such as air or water. A low drag coefficient indicates the object will have less aerodynamic or hydrodynamic drag. The lateral lift coefficient and the vertical lift coefficient denoted by C_y and C_z, respectively, are dimensionless coefficients that relates the lift generated by a lifting body to the density of the fluid around the body. It is common to show, for a particular airfoil section, the relationship between section lift coefficient and angle of attack. It is also useful to show the relationship between section lift coefficient and drag coefficient.

The coefficients C_x, C_y and C_z, which are always associated with a particular surface area S, are defined [2, 15, 34] as:

$$C_x(t) = \frac{2\,F_1(\mathbf{u},\pi)}{\rho\overline{\boldsymbol{\psi}}_\infty^2 S}, \quad C_y(t) = \frac{2\,F_2(\mathbf{u},\pi)}{\rho\overline{\boldsymbol{\psi}}_\infty^2 S}, \quad C_z(t) = \frac{2\,F_3(\mathbf{u},\pi)}{\rho\overline{\boldsymbol{\psi}}_\infty^2 S}, \tag{1.4}$$

where the fluid density ρ is taken to $\rho = 1$ in the present paper and

$$F_i(\mathbf{u},\pi) = -\int_{\Gamma_c} [\nu\nabla\mathbf{u}\cdot\mathbf{n} - \pi\mathbf{n}]\cdot\mathbf{e}_i\, d\zeta, \quad i = 1,\cdots,d. \tag{1.5}$$

The control of the unsteady viscous flow past bluff bodies has been studied by a number of authors, e.g. [3, 8, 13, 16, 22] for the passive control, [1, 4, 5, 12, 23, 24, 37] for the active open-loop control, and [2, 10, 21, 25, 26, 31] for active closed-loop control, also called a feedback control. Feedback control methods are an attractive choice over passive and active open-loop controls in that the control input is continuously modified according to the response of the flow system. For more examples of control over a bluff body, one can refer to the review work of H. Choi et al [14].

In the above-mentioned papers, the authors aim at decreasing the mean drag coefficient, suppress the vortex shedding, narrow the wake width and/or to stabilize the system around a given steady-state flow. In particular, the reduction of the drag coefficient remains a difficult and challenging issue and an important question arises: what is the lowest possible drag achievable from control in the case of bluff bodies? For example, by employing a high-frequency rotation of the circular cylinder, Tokumaru and Dimotakis [37] experimentally obtained approximately 80% drag reduction at $R_e = 15\,000$. A significant drag reduction is also obtained by Amitay et al. [1], Glezer and Amitay [19], for high Reynolds numbers ranging from 31 000 to 131 000, by applying a high-frequency forcing from a synthetic jet to flow over a circular cylinder.

Apart from experimental and numerical simulations studies, a number of theoretical works have focussed about the stabilisation around a prescribed equilibrium state, e.g. [6, 7, 17, 29, 30, 32, 33]. In most of these theoretical stabilization results, and thanks to the employed control laws, the authors aim to suppress the vortex shedding and narrow the wake width. Further, in [29] (in finite dimension) and in [32] (in infinite dimension), the stabilization result is obtained via enough small initial perturbations. However, if the above-mentioned studies aim to find an equilibrium state, such an equilibrium state is not reached by prescribing the drag coefficent C_x and the lift coefficents C_y and C_z.

This is why the present paper aim to present a theoretical study regarding the feedback control over a bluff body for prescribed drag and lift coefficients (which can be as small as desired). To our knowledge, such a study has not been conducted previously, and it is the main objective of the present paper.

For prescribed time functions $\widetilde{\lambda}_i(t),\ i = 1, \cdots, d$, we need to find a feedback control $\mathbf{v}_c = \mathcal{M}(\mathbf{u})$, where $\mathcal{M}$ is the feedback law, such that F_i in (1.5) satisfies

$$F_i(\mathbf{u}, \pi) = \widetilde{\lambda}_i(t),\ i = 1, \cdots, d. \tag{1.6}$$

To this end, the boundary control $\mathbf{v}_c$ in (1.2) is written on the form

$$\mathbf{v}_c(t, \mathbf{x}) = \sum_{i=1}^{d} \alpha_i(t)\, \mathbf{e}_i(\mathbf{x}) \qquad \text{on } \Sigma_c, \tag{1.7}$$

where the quantities $\alpha_i,\ i = 1, \cdots, d$, are a priori unknown and have to be determined in the feedback form. In order to determine α_i, leading to the determination of the boundary control $\mathbf{v}_c$, we consider the trajectory $(\boldsymbol{\psi}, q) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ solution of the stationary Navier-Stokes model [18]:

$$\begin{cases} -\nu\Delta\, \boldsymbol{\psi} + (\boldsymbol{\psi} \cdot \nabla)\boldsymbol{\psi} + \nabla q = \mathbf{f}, & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{\psi} = 0 & \text{in } \Omega, \\ \boldsymbol{\psi} = 0 & \text{on } \Gamma_c, \\ \boldsymbol{\psi} = \boldsymbol{\psi}_\infty & \text{on } \Gamma_b, \end{cases} \tag{1.8}$$

and we substitute $(\mathbf{u}, \pi)$ by $(\mathbf{v} + \boldsymbol{\psi}, p + q)$ in (1.2) and (1.6). Consequently, we get this extended system which is considered in the following

$$\begin{cases} (a)\ \ \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v} \cdot \nabla)\boldsymbol{\psi} + (\boldsymbol{\psi} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0 & \text{in } Q, \\ (b)\ \ \nabla \cdot\ \mathbf{v} = 0 & \text{in } Q, \\ (c)\ \ \mathbf{v} = 0 & \text{on } \Sigma_b, \\ (d)\ \ \mathbf{v} = \sum_{i=1}^{d} \alpha_i(t)\, \mathbf{e}_i(\mathbf{x}) & \text{on } \Sigma_c, \\ (e)\ \ \mathbf{v}(t = 0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) - \boldsymbol{\psi}(\mathbf{x}) & \text{in } \Omega, \\ (f)\ \ \displaystyle\int_{\Gamma_c} [\nu\nabla\mathbf{v} \cdot \mathbf{n} - p\mathbf{n}] \cdot \mathbf{e}_i\, d\zeta = \lambda_i(t),\ i = 1, \cdots, d, & \end{cases} \tag{1.9}$$

where $\lambda_i(t) = -F_i(\boldsymbol{\psi}, q) - \widetilde{\lambda}_i(t),\ i = 1, \cdots, d$. As in [29, 30] where the authors stabilize the two and three-dimensional Navier-Stokes problem around a given stationary state, system (1.9) is solved via a Galerkin procedure. Such a procedure consists on building a sequence of approximated solutions using an adequate Galerkin basis.

The paper is organized as follows. In section 2, the notations and mathematical preliminaries are given. In section 3, the existence of at least one solution of the non-linear extended system (1.9) is established by applying the Galerkin method.

2. Notation and Preliminaries

2.1. Function Spaces

The usual function spaces $L^2(\Omega)$, $H^1(\Omega)$, $H^1_0(\Omega)$ are used and we let $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}^1(\Omega) = (H^1(\Omega))^d$, $\mathbf{H}^1_0(\Omega) = (H^1_0(\Omega))^d$. Further, we denote by $\|\cdot\| = \|\cdot\|_{\mathbf{L}^2(\Omega)}$ the norm in $\mathbf{L}^2(\Omega)$. Finaly, if $\mathbf{u} \in \mathbf{L}^2(\Omega)$ is such that $\nabla \cdot \mathbf{u} \in L^2(\Omega)$, the normal trace of $\mathbf{u}$ in $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ is $\mathbf{u} \cdot \mathbf{n}$.

A few spaces are now introduced:

$$\mathbf{V}^1(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) \ :\ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega\}, \tag{2.1}$$

$$\mathbf{V}^1_0(\Omega) = \{\mathbf{u} \in \mathbf{H}^1_0(\Omega) \ :\ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega\}, \tag{2.2}$$

$$\mathbf{V}(\Omega) = \{\mathbf{u} \in \mathbf{V}^1(\Omega),\ \mathbf{u} = 0 \text{ on } \Gamma_b, \int_{\Gamma_c} \mathbf{u} \cdot \mathbf{n}\, d\zeta = 0\}, \tag{2.3}$$

$$\mathbf{H}(\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) \ :\ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_b\}. \tag{2.4}$$

$\mathbf{H}(\Omega)$ is a Hilbert space endowed with $\mathbf{L}^2$-norm and $\mathbf{V}(\Omega)$ is Hilbert space endowed with $\mathbf{H}^1$-norm. Denoting by $\mathbf{V}^{-1}(\Omega) = (\mathbf{V}^1_0(\Omega))'$ the dual space of $\mathbf{V}^1_0(\Omega)$ and considering $\mathbf{H}(\Omega)$ identified with its own dual, we have $\mathbf{V}(\Omega) \subset \mathbf{H}(\Omega) \subset \mathbf{V}^{-1}(\Omega)$ algebraically and topologically with compact injections.

Finally, the solution $\mathbf{v}$ of (1.9) is searched in the space

$$\mathbf{W}(\Omega) = \{\mathbf{v} \in \mathbf{V}(\Omega), \exists\, \boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d) \text{ such that } \mathbf{v} = \sum_{i=1}^{d} \alpha_i \mathbf{e}_i \text{ on } \Gamma_c\}, \tag{2.5}$$

where the orthonormal basis $\mathbf{e}_i$ of $\mathbb{R}^3$ is such that $\mathbf{e}_i \in \mathbf{V}^{1/2}(\Gamma_c)$, $i = 1, \cdots, d$.

2.2. Linear Forms and a few inequalities

In order to define a weak form of the Navier-Stokes equations, we introduce the continuous bilinear form

$$a(\mathbf{v}_1, \mathbf{v}_2) = \int_\Omega \nabla \mathbf{v}_1 : \nabla \mathbf{v}_2 \, d\mathbf{x}, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{H}^1(\Omega),$$

and the trilinear form

$$b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \int_\Omega (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_2 \cdot \mathbf{v}_3 \, d\mathbf{x}, \quad \forall \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{H}^1(\Omega).$$

Thanks to Hölder inequality, we obtain

$$|b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)| \leq \|\mathbf{v}_1\|_{\mathbf{L}^3(\Omega)} \|\nabla \mathbf{v}_2\| \|\mathbf{v}_3\|_{\mathbf{L}^6(\Omega)}.$$

Further, due to the generalized Sobolev's inequality, there exists a positive constant C such that

$$\|\mathbf{v}_1\|_{\mathbf{L}^3(\Omega)} \le C\|\mathbf{v}_1\|^{\frac{1}{2}}\|\nabla\mathbf{v}_1\|^{\frac{1}{2}} \text{ and } \|\mathbf{v}_3\|_{\mathbf{L}^6(\Omega)} \le C\|\nabla\mathbf{v}_3\|, \text{ for } d=2,3,$$

and hence,

$$|b(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)| \le C\|\mathbf{v}_1\|^{\frac{1}{2}}\|\nabla\mathbf{v}_1\|^{\frac{1}{2}}\|\nabla\mathbf{v}_2\|\|\nabla\mathbf{v}_3\|. \tag{2.6}$$

By using Hölder inequality, we obtain

$$|b(\mathbf{v},\mathbf{u},\mathbf{v})| \le \|\mathbf{v}\|^2_{\mathbf{L}^4(\Omega)}\|\nabla\mathbf{u}\|, \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega),$$

and hence thanks to [11, Remark III.2.17], we deduce

$$|b(\mathbf{v},\mathbf{u},\mathbf{v})| \le C\|\mathbf{v}\|^{2-\frac{d}{2}}\|\nabla\mathbf{v}\|^{\frac{d}{2}}\|\nabla\mathbf{u}\|, \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega). \tag{2.7}$$

By employing integration by parts, the following property holds true

$$b(\mathbf{u},\mathbf{v},\mathbf{v}) = \frac{1}{2}\int_{\Gamma_c} |\mathbf{v}|^2(\mathbf{u}\cdot\mathbf{n})\,d\zeta, \quad \forall \mathbf{u} \in \mathbf{V}^1(\Omega) \text{ and } \forall \mathbf{v} \in \mathbf{V}(\Omega). \tag{2.8}$$

For all $\mathbf{v} = \sum_{i=1}^d \alpha_i \mathbf{e}_i$ and $\widetilde{\mathbf{v}} = \sum_{i=1}^d \widetilde{\alpha}_i \mathbf{e}_i$ on Γ_c, we have

$$\mathbf{v}\cdot\widetilde{\mathbf{v}} = \sum_{i=1}^d \alpha_i\widetilde{\alpha}_i \text{ on } \Gamma_c \quad \text{ and } \quad \mathbf{v}\cdot\mathbf{n} = \sum_{i=j}^d \alpha_j(\mathbf{e}_j\cdot\mathbf{n}) \text{ on } \Gamma_c. \tag{2.9}$$

From the trace theorem and the Poincarré inequality, we obtain $\|\mathbf{v}\|_{\mathbf{L}^2(\Gamma)} \le C\|\nabla\mathbf{v}\|$, $\forall \mathbf{v} \in \mathbf{W}(\Omega)$, and hence

$$\|\mathbf{v}\|_{\mathbf{L}^2(\Gamma_c)} = \sqrt{\sum_{i=1}^d \alpha_i^2} \le C\|\nabla\mathbf{v}\|. \tag{2.10}$$

In the next Section, the variational formulation of the stabilization problem (1.9) is given.

3. Existence Result

3.1. The variational formulation

We consider the variational formulation for the extended system (1.9).

Definition 3.1. Let $T > 0$ be an arbitrary real number, $\lambda_i(t)$ in $L^2(0,T)$, $i = 1,\cdots,d$ and $\mathbf{v}_0 \in \mathbf{H}(\Omega)$, we shall say that $\mathbf{v}$ is a weak solution of (1.9) on $[0,T)$ if

- $\mathbf{v} \in L^\infty(0,T;\mathbf{H}(\Omega)) \cap L^2(0,T;\mathbf{V}(\Omega))$,
- $\exists\, \boldsymbol{\alpha} = (\alpha_1,\cdots,\alpha_d) \in \mathbf{L}^2(0,T)$ such that $\mathbf{v} = \sum_{i=1}^d \alpha_i\,\mathbf{e}_i$ on Γ_c,

$$\begin{cases} (a) & \dfrac{d}{dt}\displaystyle\int_\Omega \mathbf{v}\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} + \nu a(\mathbf{v},\widetilde{\mathbf{v}}) + b(\mathbf{v},\boldsymbol{\psi},\widetilde{\mathbf{v}}) + b(\boldsymbol{\psi},\mathbf{v},\widetilde{\mathbf{v}}) + b(\mathbf{v},\mathbf{v},\widetilde{\mathbf{v}}) = \sum_{i=1}^d \widetilde{\alpha}_i\lambda_i, \\ (b) & \left(\displaystyle\int_\Omega \mathbf{v}\cdot\tilde{\mathbf{v}}\;d\mathbf{x}\right)(0) = \displaystyle\int_\Omega \mathbf{v}_0\cdot\tilde{\mathbf{v}}\;d\mathbf{x}, \end{cases} \tag{3.1}$$

$\forall\, \widetilde{\mathbf{v}} \in \mathbf{W}(\Omega)$ with $\widetilde{\mathbf{v}} = \sum_{i=1}^d \widetilde{\alpha}_i\,\mathbf{e}_i$ on Γ_c.

Note that the initial condition $(3.1)_b$ makes sense because for any solution $\mathbf{v}$ of $(3.1)_a$, function $t \to \int_\Omega \mathbf{v}(t) \cdot \widetilde{\mathbf{v}}\, d\mathbf{x}$ is continuous (see [11] Corollaire II.4.2).

We now first establish the a priori estimates for the extended system (1.9).

3.2. A priori estimates

Taking $\widetilde{\mathbf{v}} = \mathbf{v}$ in $(3.1)_a$ leads to

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\|\nabla\mathbf{v}\|^2 + b(\mathbf{v},\mathbf{v},\mathbf{v}) + b(\boldsymbol{\psi},\mathbf{v},\mathbf{v}) + b(\mathbf{v},\boldsymbol{\psi},\mathbf{v}) = \sum_{i=1}^{d}\alpha_i\lambda_i. \tag{3.2}$$

Firstly, let us estimate the terms of $b(\cdot,\cdot,\cdot)$ in (3.2). Using (2.8), yields

$$b(\mathbf{v},\mathbf{v},\mathbf{v}) = \frac{1}{2}\int_{\Gamma_c}|\mathbf{v}|^2(\mathbf{v}\cdot\mathbf{n})\,d\zeta, \quad \forall \mathbf{v} \in \mathbf{W}(\Omega). \tag{3.3}$$

Using (1.1) and (2.9) in (3.3), we obtain

$$b(\mathbf{v},\mathbf{v},\mathbf{v}) = \frac{1}{2}\left(\sum_{i=1}^{d}\alpha_i^2\right)\sum_{j=1}^{d}\alpha_j\int_{\Gamma_c}\mathbf{e}_j\cdot\mathbf{n}\,d\zeta = 0. \tag{3.4}$$

Secondly, from (2.8), we have

$$b(\boldsymbol{\psi},\mathbf{v},\mathbf{v}) = \frac{1}{2}\int_{\Gamma_c}|\mathbf{v}|^2(\boldsymbol{\psi}\cdot\mathbf{n})\,d\zeta, \quad \forall \mathbf{v} \in \mathbf{W}(\Omega),$$

and since $\boldsymbol{\psi} = 0$ on Γ_c, we deduce

$$b(\boldsymbol{\psi},\mathbf{v},\mathbf{v}) = 0. \tag{3.5}$$

Finally, using (2.6) and Young's inequality leads to

$$\begin{aligned}
|b(\mathbf{v},\boldsymbol{\psi},\mathbf{v})| &\leq C\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\mathbf{v}\|^{\frac{1}{2}}\|\nabla\boldsymbol{\psi}\|\|\nabla\mathbf{v}\| \\
&\leq \frac{\epsilon_1}{2}\|\nabla\mathbf{v}\|^2 + \frac{C^2}{2\epsilon_1}\|\nabla\boldsymbol{\psi}\|^2\|\mathbf{v}\|\|\nabla\mathbf{v}\| \\
&\leq \frac{\epsilon_1+\epsilon_2}{2}\|\nabla\mathbf{v}\|^2 + \frac{1}{2\epsilon_2}\left(\frac{C^4}{4\epsilon_1^2}\|\nabla\boldsymbol{\psi}\|^4\right)\|\mathbf{v}\|^2
\end{aligned}$$

and by taking $\epsilon_1 = \epsilon_2 = \dfrac{\nu}{4}$, we obtain

$$|b(\mathbf{v},\boldsymbol{\psi},\mathbf{v})| \leq \frac{\nu}{4}\|\nabla\mathbf{v}\|^2 + \left(\frac{8C^4}{\nu^3}\|\nabla\boldsymbol{\psi}\|^4\right)\|\mathbf{v}\|^2. \tag{3.6}$$

We now estimate the term in the right side of (3.2). Using (2.10), we obtain $|\alpha_i| \leq C\|\nabla\mathbf{v}\|$ and hence

$$\sum_{i=1}^{d}\alpha_i\lambda_i \leq C\,\|\nabla\mathbf{v}\|\left(\sum_{i=1}^{d}|\lambda_i|\right) \leq \frac{\nu}{4}\|\nabla\mathbf{v}\|^2 + M_\lambda(t) \tag{3.7}$$

where $M_\lambda(t) = \frac{1}{\nu}\left(C\sum_{i=1}^{d}|\lambda_i(t)|\right)^2$. Using (3.4)-(3.7) in (3.2), the following inequality holds

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \frac{\nu}{2}\|\nabla\mathbf{v}\|^2 \le \left(\frac{8C^4}{\nu^3}\|\nabla\boldsymbol{\psi}\|^4\right)\|\mathbf{v}\|^2 + M_\lambda(t). \tag{3.8}$$

Consequently, thanks to Gronwall lemma, we deduce from (3.8) the following estimation

$$\sup_{t\le T}\|\mathbf{v}(t)\|^2 + \nu\int_0^T \|\nabla\mathbf{v}(t)\|^2 dt \le C_\lambda(T), \tag{3.9}$$

where $C_\lambda(T)$ depends on T, M_λ, $\|\nabla\boldsymbol{\psi}\|$ and $\|\mathbf{v}_0\|$.

Let us estimate $\frac{d\mathbf{v}}{dt}$. By using integration by parts and the technics used in (3.4)-(3.5), we show that

$$\begin{aligned} b(\mathbf{v},\boldsymbol{\psi},\widetilde{\mathbf{v}}) &= -b(\mathbf{v},\widetilde{\mathbf{v}},\boldsymbol{\psi}),\\ b(\mathbf{v},\mathbf{v},\widetilde{\mathbf{v}}) &= -b(\mathbf{v},\widetilde{\mathbf{v}},\mathbf{v}). \end{aligned}$$

Moreover, by employing (2.6) and (2.7), we obtain

$$\begin{aligned} |b(\boldsymbol{\psi},\mathbf{v},\widetilde{\mathbf{v}})| &\le C\|\boldsymbol{\psi}\|^{\frac{1}{2}}\|\nabla\boldsymbol{\psi}\|^{\frac{1}{2}}\|\nabla\mathbf{v}\|\|\nabla\widetilde{\mathbf{v}}\|,\\ |b(\mathbf{v},\widetilde{\mathbf{v}},\boldsymbol{\psi})| &\le C\|\boldsymbol{\psi}\|^{\frac{1}{2}}\|\nabla\boldsymbol{\psi}\|^{\frac{1}{2}}\|\nabla\mathbf{v}\|\|\nabla\widetilde{\mathbf{v}}\|,\\ |b(\mathbf{v},\widetilde{\mathbf{v}},\mathbf{v})| &\le C\|\mathbf{v}\|^{2-\frac{d}{2}}\|\nabla\mathbf{v}\|^{\frac{d}{2}}\|\nabla\widetilde{\mathbf{v}}\|, \end{aligned}$$

hence, from (3.1), by taking $\widetilde{\alpha}_i = 0$, yielding $\widetilde{\mathbf{v}} \in \mathbf{V}_0^1(\Omega)$, we deduce

$$\left\|\frac{d\mathbf{v}}{dt}\right\|_{\mathbf{V}^{-1}(\Omega)} \le \nu\|\nabla\mathbf{v}\| + C\|\boldsymbol{\psi}\|^{\frac{1}{2}}\|\nabla\boldsymbol{\psi}\|^{\frac{1}{2}}\|\nabla\mathbf{v}\| + C\|\mathbf{v}\|^{2-\frac{d}{2}}\|\nabla\mathbf{v}\|^{\frac{d}{2}} := \mathcal{G}(t),$$

where $\mathcal{G}(t)$ is bounded in $L^{\frac{4}{d}}(]0,T[)$ according to estimate (3.9). Therefore,

$$\left\|\frac{d\mathbf{v}}{dt}\right\|_{L^{\frac{4}{d}}(]0,T[;\mathbf{V}^{-1}(\Omega))} \le \left(\int_0^T \mathcal{G}^{\frac{4}{d}}(t)\,dt\right)^{\frac{d}{4}} \le C_\lambda(T). \tag{3.10}$$

Theorem 3.2. *Assume that the bluff body whose boundary is Γ_c, is such that*

$$\int_{\Gamma_c} \mathbf{e}_i\cdot\mathbf{n}\,d\zeta = 0, \quad \mathbf{e}_i = (\delta_{1,i},\delta_{2,i},\delta_{3,i}),\ i =, 1\cdots, d \tag{3.11}$$

with $\delta_{i,j}$ the Kronecker symbol.

For an arbitrary function λ_i in $L^2(0,T)$, $i = 1,\cdots,d$ and an arbitrary initial data $\mathbf{v}_0$ in $\mathbf{H}(\Omega)$, there exists a solution $\mathbf{v}$ in the sense of definition 3.1, and a distribution p on Q such that (1.9) holds. Moreover, $\frac{d\mathbf{v}}{dt}$ belongs to $L^{\frac{d}{4}}(]0,T[;\mathbf{V}^{-1}(\Omega))$.

Proof. In the first step a Galerkin basis is built for the space $\mathbf{W}(\Omega)$ defined in (2.5) while in the second step we prove the existence of a weak solution $\mathbf{v}$. Finally, we prove the existence of the pressure.

3.3. *A Galerkin basis for the space* $\mathbf{W}(\Omega)$

For $i = 1, \cdots, d$, we consider the following Stokes problem

$$\begin{cases} -\Delta \mathbf{w}_i + \nabla q_i = 0, & \text{in } \Omega, \\ \nabla \cdot \mathbf{w}_i = 0 & \text{in } \Omega, \\ \mathbf{w}_i = 0 & \text{on } \Gamma_b, \\ \mathbf{w}_i = \mathbf{e}_i & \text{on } \Gamma_c. \end{cases} \tag{3.12}$$

From condition (1.1), $\int_{\Gamma_c} \mathbf{e}_i \cdot \mathbf{n} \, d\zeta = 0$. Thus, system (3.12) admits a unique solution $(\mathbf{w}_i, q_i)$ belonging to $\mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ (see [11, Theorem IV.6.5]). Moreover, for all $\mathbf{z} \in \mathbf{V}^1_0(\Omega)$ defined in (2.2) and for all $\alpha_i \in \mathbb{R}$, we have $\mathbf{v} = \mathbf{z} + \sum_{i=1}^d \alpha_i \mathbf{w}_i \in \mathbf{W}(\Omega)$, where $\mathbf{w}_i$ satisfies (3.12). Indeed, we have $\mathbf{z}, \mathbf{w}_i \in \mathbf{V}(\Omega)$ and since $\mathbf{z} = 0$ on Γ_c, we obtain $\mathbf{v} = \sum_{i=1}^d \alpha_i \mathbf{w}_i$ on Γ_c. When $(\mathbf{z}_n)_{n\in\mathbb{N}}$ defines a countable orthonormal basis of $\mathbf{V}^1_0(\Omega)$, since $\mathbf{w}_i = \mathbf{e}_i$ on Γ_c, the sequence $\mathbf{w}_1, \cdots, \mathbf{w}_d, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \cdots$, is then linearly independent. Consequently, $\mathbf{W}(\Omega)$ can be rewritten as

$$\mathbf{W}(\Omega) = \operatorname{span}(\mathbf{w}_i)_{\{1\le i\le d\}} \oplus \operatorname{span}(\mathbf{z}_n)_{\{n\in\mathbb{N}^*\}}, \tag{3.13}$$

and $\mathbf{v}$ is expressed as:

$$\mathbf{v} = \mathbf{z} + \sum_{i=1}^{d} \alpha_i \mathbf{w}_i, \qquad \text{with} \qquad \mathbf{z} = \sum_{i=1}^{\infty} \theta_i \mathbf{z}_i.$$

3.4. *Existence of weak solution*

The proof of the existence follows a standard procedure [30] . In a first step a sequence of approximate solutions using a Galerkin method is built. A compactness result allows us to pass to the limit in the system satisfied by the approximated solutions.

3.4.1. *The Galerkin Method*

Let $m \in \mathbb{N}^*$, we define the space

$$W_m = \operatorname{span}(\mathbf{w})_{\{1\le i\le d\}} \oplus \operatorname{span}(\mathbf{z}_i)_{\{1\le i\le m\}}$$

and we express $\mathbf{v}_m \in W_m$ as:

$$\mathbf{v}_m = \sum_{i=1}^{d+m} \alpha_{im} \boldsymbol{\varphi}_i,$$

where $\boldsymbol{\varphi}_i = \mathbf{w}_i$, for $i = 1, \cdots, d$ and $\boldsymbol{\varphi}_i = \mathbf{z}_{i-d}$ for $i = d+1, d+2, \cdots, d+m$. We consider the following finite-dimensional problem

$$\begin{cases} (a) \quad \dfrac{d}{dt} \displaystyle\int_\Omega \mathbf{v}_m \cdot \boldsymbol{\varphi}_j \, d\mathbf{x} + \nu a(\mathbf{v}_m, \boldsymbol{\varphi}_j) + b(\mathbf{v}_m, \boldsymbol{\psi}, \boldsymbol{\varphi}_j) + b(\boldsymbol{\psi}, \mathbf{v}_m, \boldsymbol{\varphi}_j) \\ \qquad\qquad + b(\mathbf{v}_m, \mathbf{v}_m, \boldsymbol{\varphi}_j) = \displaystyle\sum_{i=1}^{d} \delta_{ij} \lambda_i, \\ (b) \quad \displaystyle\int_\Omega \mathbf{v}_m(0) \cdot \boldsymbol{\varphi}_j \, d\mathbf{x} = \int_\Omega \mathbf{v}_0 \cdot \boldsymbol{\varphi}_j \, d\mathbf{x}, \quad \text{for } j = 1, 2, \cdots, d+m. \end{cases} \tag{3.14}$$

Lemma 3.3. *The discrete problem (3.14) has a unique solution* $\mathbf{v}_m$ *belonging to* $C^1(0, T_m; W_m)$. *Moreover the solution satisfies :*

$$\|\mathbf{v}_m\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{v}_m\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \leq C_\lambda(T), \tag{3.15}$$

$$\left\|\frac{d\mathbf{v}_m}{dt}\right\|_{L^{\frac{4}{d}}(]0,T[;\mathbf{V}^{-1}(\Omega))} \leq C_\lambda(T), \tag{3.16}$$

where $C_\lambda(T)$ *is a positive constant independent of* m.

Proof. Classical results of nonlinear ODEs lead to the existence of the greatest T_m in $(0, T)$ such that the discrete problem (3.14) has a unique solution $\mathbf{v}_m \in C^1(0, T_m; W_m)$. Indeed, the resulting mass matrix defined as $M_{ij} = \int_\Omega \boldsymbol{\varphi}_i \cdot \boldsymbol{\varphi}_j \, d\mathbf{x}$ $(1 \leq i, j \leq d + m)$ is nonsingular. In order to show that T_m is independent of m, it is sufficient to verify the boundedness of the L^2-norm of $\mathbf{v}_m$ independently of m. Following the same procedure as for the derivation of the a priori estimates (3.9) and (3.10) yield (3.15) and (3.16). If $T_m < T$, then $\|\mathbf{v}_m\|$ should tend to $+\infty$ as $t \to T_m$ because of the explosion criteria. However, this does not happen since $\|\mathbf{v}_m\|$ is bounded independently of m in (3.15), and therefore $T_m = T$. □

For a subsequence of $\mathbf{v}_m$ (still denoted by $\mathbf{v}_m$), the estimates in (3.15) and (3.16) yield the following weak convergences as m tends to ∞ :

$$\begin{cases} \mathbf{v}_m \rightharpoonup \mathbf{v} \text{ weakly in } L^2(]0,T[;\mathbf{V}(\Omega)), \\ \mathbf{v}_m \rightharpoonup \mathbf{v} \text{ weakly* in } L^\infty(]0,T[;\mathbf{H}(\Omega)), \\ \dfrac{d\mathbf{v}_m}{dt} \rightharpoonup \dfrac{d\mathbf{v}}{dt} \text{ weakly in } L^{\frac{4}{d}}(]0,T[;\mathbf{V}^{-1}(\Omega)). \end{cases} \tag{3.17}$$

Nevertheless, the convergences in (3.17) are not sufficient to pass to the limit in the weak formulation (3.14), because of the presence of the convection term. Consequently, in order to utilize the compactness theory on the sequence of approximated solution $\mathbf{v}_m$, we need to apply the Aubin theorem [27, Théorème 5.1, page 58] with $p_0 = 2$, $p_1 = \frac{4}{d}$ and $B_0 = \mathbf{V}(\Omega)$, $B_1 = \mathbf{V}^{-1}(\Omega)$ and $B = \mathbf{H}(\Omega)$. Note that $B_0 \subset B \subset B_1$, and the imbedding from B_0 to B is compact. Wet set

$$\boldsymbol{U} = \{\mathbf{v}, \mathbf{v} \in L^2(]0,T[;\mathbf{V}(\Omega)), \mathbf{v} \in L^{\frac{4}{d}}(]0,T[;\mathbf{V}^{-1}(\Omega))\}$$

and with the norm $\|\mathbf{v}\|_{L^2(]0,T[;\mathbf{V}(\Omega))} + \|\mathbf{v}\|_{L^{\frac{4}{d}}(]0,T[;\mathbf{V}^{-1}(\Omega))}$, $\boldsymbol{U}$ is a Banach space. Then by applying the Aubin compacity theorem we prove that the imbedding $\boldsymbol{U} \subset L^2(]0,T[;\mathbf{H}(\Omega))$ is compact; and hence we obtain the following strong convergence (at least for a subsequence of $\mathbf{v}_m$ still denoted by $\mathbf{v}_m$)

$$\mathbf{v}_m \to \mathbf{v} \text{ strongly in } L^2(0,T;\mathbf{L}^2(\Omega)). \tag{3.18}$$

Using the above strong convergence result and (3.17) enable us to pass to the limit in the weak formulation, obtained from (3.14) after multiplication by $\varphi \in \mathcal{D}([0,T))$ and integration by parts with respect to time. Hence, for all $\widetilde{\mathbf{v}}_j = \widetilde{\alpha}_j \boldsymbol{\varphi}_j,\ j = 1, 2, 3, \cdots, d+m$, passing to the limit yields

$$\begin{aligned}
&-\int_0^T \int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}}_j \varphi'(t)\, d\mathbf{x}\, dt + \int_\Omega \mathbf{v}_0 \widetilde{\mathbf{v}}_j \varphi(0)\, d\mathbf{x} + \nu \int_0^T \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}\, dt \\
&\quad + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}\, dt + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \boldsymbol{\psi}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}\, dt \\
&\quad\quad + \int_0^T \int_\Omega (\boldsymbol{\psi} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}\, dt = \int_0^T \widetilde{\alpha}_j \delta_{jk} \lambda_k(t) \varphi(t)\ dt,
\end{aligned} \tag{3.19}$$

By linearity, equation (3.19) holds for all $\widetilde{\mathbf{v}}$ combination of finite $\widetilde{\mathbf{v}}_j$ and by density, for any element of $\mathbf{W}(\Omega)$. Finally, by using (2.10) and the second term of (3.15), we show that $\boldsymbol{\alpha}_m = (\alpha_{1m}, \cdots, \alpha_{dm})$ belongs to $\mathbf{L}^2(0,T)$. Then, for a subsequence of α_{im} (still denoted by α_{im}) we have

$$\alpha_{im} \rightharpoonup \alpha_i \text{ weakly in } L^2(0,T), \quad i = 1, \cdots, d. \qquad \square$$

Now we can retrieve the stabilized problem (1.9).

3.5. Retrieving the stabilized problem

First, we prove the existence of the pressure.

Lemma 3.4. *There exists $p \in \mathcal{D}'(]0,T[; L^2(\Omega))$ such that $(\mathbf{v}, p)$ satisfies $(1.9)_a$ in the distribution sense.*

Proof. By choosing $\varphi \in \mathcal{D}(0,T)$ in (3.19), $\forall \widetilde{\mathbf{v}} = \widetilde{\mathbf{z}} + \widetilde{\alpha}_j \mathbf{w}_j \in \mathbf{W}(\Omega),\ j = 1, \cdots, d$ and $\widetilde{\mathbf{z}} \in \mathbf{V}_0^1(\Omega)$, we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \widetilde{\mathbf{v}}\, \varphi(t)\, d\mathbf{x}\, dt + \nu \int_0^T \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}}\, \varphi(t)\, d\mathbf{x}\, dt \\
&\quad + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}\, \varphi(t)\, d\mathbf{x}\, dt + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \boldsymbol{\psi}) \cdot \widetilde{\mathbf{v}}\, \varphi(t)\, d\mathbf{x}\, dt \\
&\quad\quad + \int_0^T \int_\Omega (\boldsymbol{\psi} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}\, \varphi(t)\, d\mathbf{x}\, dt = \int_0^T \widetilde{\alpha}_j \lambda_j(t)\, \varphi(t) dt,
\end{aligned} \tag{3.20}$$

hence

$$\begin{aligned}
&\int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \widetilde{\mathbf{v}}\, d\mathbf{x} + \nu \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}}\, d\mathbf{x} + \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}\, d\mathbf{x} + \int_\Omega (\mathbf{v} \cdot \nabla \boldsymbol{\psi}) \cdot \widetilde{\mathbf{v}}\, d\mathbf{x} \\
&\quad + \int_\Omega (\boldsymbol{\psi} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}\, d\mathbf{x} = \widetilde{\alpha}_i \lambda_i(t) \quad \text{in } \mathcal{D}'(0,T).
\end{aligned} \tag{3.21}$$

Further, taking $\widetilde{\alpha}_i = 0,\ i = 1, \cdots, d$, yielding $\widetilde{\mathbf{v}} \in \mathbf{V}_0^1(\Omega)$, we deduce

$$\begin{aligned}
&\int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \widetilde{\mathbf{v}}\, d\mathbf{x} + \nu \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}}\, d\mathbf{x} + \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}\, d\mathbf{x} \\
&\quad + \int_\Omega (\mathbf{v} \cdot \nabla \boldsymbol{\psi}) \cdot \widetilde{\mathbf{v}}\ d\mathbf{x} + \int_\Omega (\boldsymbol{\psi} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}\, d\mathbf{x} = 0 \quad \text{in } \mathcal{D}'(0,T).
\end{aligned} \tag{3.22}$$

Then, letting

$$\mathbf{f} = \frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \boldsymbol{\psi} + (\boldsymbol{\psi} \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}$$

and using (3.22) leads to $\mathbf{f} \in \mathcal{D}'(]0,T[\,;\,\mathbf{H}^{-1}(\Omega))$ and $\langle \mathbf{f}, \widetilde{\mathbf{v}}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} = 0$, for all $\widetilde{\mathbf{v}}$ in $\mathbf{V}_0^1(\Omega)$. Hence, due to de Rham's theorem[36], there exists $p \in \mathcal{D}'(]0,T[\,;\,L^2(\Omega))$ such that $\mathbf{f} = -\nabla p$. $\square$

Next, we prove that $(\mathbf{v}, p)$ satisfies $(1.9)_f$. By writing $(1.9)_a$ in the form

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\cdot(-\nu\nabla\mathbf{v} + Ip) + (\mathbf{v}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = 0 \ \text{ in } Q$$

and using [36, Chap I, Theorem 1.2], we obtain

$$\begin{aligned}\int_\Omega \frac{\partial \mathbf{v}}{\partial t}\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} + \int_\Omega(\nu\nabla\mathbf{v} - Ip) : \nabla\widetilde{\mathbf{v}}\,d\mathbf{x} + \langle(-\nu\nabla\mathbf{v} + Ip)\cdot\mathbf{n}, \widetilde{\mathbf{v}}\rangle \\ + \int_\Omega(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} + \int_\Omega(\mathbf{v}\cdot\nabla\boldsymbol{\psi})\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} + \int_\Omega(\boldsymbol{\psi}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} = 0,\end{aligned}$$

for all $\widetilde{\mathbf{v}}$ in $\mathbf{W}(\Omega)$. Letting $\widetilde{\mathbf{v}} = \widetilde{\alpha}_i\mathbf{w}_i$, $i = 1, \cdots, d$, yields

$$\begin{aligned} pI : \nabla\widetilde{\mathbf{v}} = p\nabla\cdot\widetilde{\mathbf{v}} &= 0, \\ \langle(-\nu\nabla\mathbf{v} + Ip)\cdot\mathbf{n}, \widetilde{\mathbf{v}}\rangle_{\mathbf{H}^{-\frac{1}{2}}(\Gamma),\mathbf{H}^{\frac{1}{2}}(\Gamma)} &= -\widetilde{\alpha}_i\int_{\Gamma_c}[\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{e}_i\,d\zeta.\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{d}{dt}\int_\Omega \mathbf{v}\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} + \nu\int_\Omega\nabla\mathbf{v} : \nabla\widetilde{\mathbf{v}}\,d\mathbf{x} + \int_\Omega(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} + \int_\Omega(\mathbf{v}\cdot\nabla\boldsymbol{\psi})\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} \\ + \int_\Omega(\boldsymbol{\psi}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} = \widetilde{\alpha}_i\int_{\Gamma_c}[\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{e}_i\,d\zeta.\end{aligned} \tag{3.23}$$

By comparing (3.21) and (3.23), we retreive $(1.9)_f$, namely

$$\int_{\Gamma_c}[\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{e}_i\,d\zeta = \lambda_i.$$

Finally, it remains to verify that the initial condition $(1.9)_e$ belongs to $\mathbf{W}'(\Omega)$. In this purpose, we multiply $(1.9)_a$ by $\widetilde{\mathbf{v}}\varphi$, with $\varphi(T) = 0$, and integrate with respect to time and space

$$\begin{aligned}-\int_0^T\int_\Omega \mathbf{v}\cdot\widetilde{\mathbf{v}}\,\varphi'(t)\,d\mathbf{x}\,dt + \int_\Omega\mathbf{v}(0)\widetilde{\mathbf{v}}\,\varphi(0)\,d\mathbf{x} + \nu\int_0^T\int_\Omega\nabla\mathbf{v} : \nabla\widetilde{\mathbf{v}}\,\varphi(t)\,d\mathbf{x}\,dt \\ + \int_0^T\int_\Omega(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\,\varphi(t)\,d\mathbf{x}\,dt + \int_0^T\int_\Omega(\mathbf{v}\cdot\nabla\boldsymbol{\psi})\cdot\widetilde{\mathbf{v}}\,\varphi(t)\,d\mathbf{x}\,dt \\ + \int_0^T\int_\Omega(\boldsymbol{\psi}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\,\varphi(t)\,d\mathbf{x}\,dt = \int_0^T\sum_{i=1}^d\widetilde{\alpha}_i\lambda_i(t)\,\varphi(t)dt.\end{aligned} \tag{3.24}$$

By comparing (3.19) and (3.24), we obtain $\int_\Omega(\mathbf{v}(0) - \mathbf{v}_0)\cdot\widetilde{\mathbf{v}}\,\varphi(0)\,d\mathbf{x} = 0$, and choosing φ such that $\varphi(0) = 1$, yields

$$\int_\Omega(\mathbf{v}(0) - \mathbf{v}_0)\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} = 0 \quad \forall\widetilde{\mathbf{v}} \in \mathbf{W}(\Omega),$$

hence, $\mathbf{v}(0) = \mathbf{v}_0$ in $\mathbf{W}'(\Omega)$. We conclude that $\mathbf{v}$ is solution of (1.9).

4. Concluding remarks

In this paper, the stabilization of the two and three-dimensional Navier-Stokes equations in a bounded domain is studied *around a prescribed drag and lift coefficients*, using a boundary feedback control. In order to determine a feedback law, an extended system coupling the Navier-Stokes equations with an equation satisfied by the control on the domain boundary is considered. We first assume that on the bluff body Σ_c (a part of the domain boundary), the trace of the fluid velocity $\mathbf{v}_c$ is a linear combination of a given velocity field represented by $\mathbf{e}_i = (\delta_{1i}, \cdots, \delta_{di})^T$, $i = 1, \cdots, d$ and the proportionality coefficient α_i, such that $\mathbf{v}_c = \sum_{i=1}^{d} \alpha_i \mathbf{e}_i$. The quantity α_i is an unknown of the problem and it is written in a feedback form. By using the Galerkin method, α_i is determined such that the Dirichlet boundary control $\mathbf{v}_c$ is satisfied on Σ_c, and the stabilizing boundary control is built. Finally, we show that the feedback control (1.6) provides stabilization of the Navier-Stokes problem *around a given drag and lift coefficients*.

References

[1] M. Amitay, B.L. Smith and A. Glezer, Aerodynamic flow control using synthetic jet technology, 36th AIAA Aerosp. Sci. Meet., Reno, Nev., AIAA Pap. No. 98-0208, 1998.

[2] L. Amodei and J.-M. Buchot, A stabilization algorithm of the NavierStokes equations based on algebraic Bernoulli equation, Numerical Linear Algebra with Applications, 19 (2012), 700–727.

[3] E. Anderson, A. Szewczyk, Effects of a splitter plate on the near wake of a circular cylinder in 2 and 3-dimensional flow configurations, Exp. Fluids, 23 (1997), 161–174.

[4] D. Arcas, L. Redekopp, Aspects of wake vortex control through base blowing/suction. Phys. Fluids, 16 (2004), 452–456.

[5] G. Artana, R. Sosa, E. Moreau, G. Touchard, Control of the near-wake flow around a circular cylinder with electro-hydrodynamic actuators, Exp. Fluids, 35 (2003), 580–588.

[6] M. Badra and T. Takahashi, *Stabilization of parabolic nonlinear systems with finite-dimensional feedback or dynamical controllers: Application to the Navier-Stokes system*, SIAM J. Control and Optimization, 49 (2011), 420–463.

[7] V. Barbu, *Stabilization of Navier-Stokes equations by oblique boundary feedback controllers*, SIAM J. Control Optimization, 50 (2012), 2288–2307.

[8] P. W. Bearman, J. K. Harvey, Control of circular cylinder flow by the use of dimples, AIAA J., 31(1993), 1753–1756.

[9] M. Bercovier, O. Pironneau, V. Sastri, Finite elements and characteristics for some parabolic-hyperbolic problems, Appl. Math. Model, 7 (1983), 89–96.

[10] M. Bergmann, L. Cordier, J. Brancher, Optimal rotary control of the cylinder wake using proper orthogonal decomposition reduced-order model. Phys. Fluids, 17 (2005), 097101:1-21.

[11] F. Boyer and P. Fabrie, Mathematical tools for the study of the incompressible Navier-Stokes equations and related models, vol. 183, Springer, 2013.

[12] J. Carberry, J. Sheridan, D. Rockwell, Controlled oscillations of a cylinder: a new wake state, J. Fluids Struct., 17 (2003), 337–343.

[13] J. Choi, W. Jeon, H. Choi, Mechanism of drag reduction by dimples on a sphere. Phys. Fluids, 18 (2006), 041702: 1–4.

[14] H. Choi, W. P. Jeon, J. Kim, Control of flow over a bluff body, Annu. Rev. Fluid Mech., 40 (2008), 113–139.

[15] L. J. Clancy, Aerodynamics, Pitman Publishing, London, 1975.

[16] C. Dalton, Y. Xu, J. C. Owen, The suppression of lift on a circular cylinder due to vortex shedding at moderate Reynolds numbers, J. Fluids Struct., 15 (2001), 617–628 .

[17] A. V. Fursikov, *Stabilization for the 3D Navier-Stokes system by feedback boundary control. Partial Differential Equations and Applications*, Discrete and Cont. Dyn. Syst., 10 (2004), 289–314.

[18] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations, Vol. II, Nonlinear steady problems, volume 39 of Springer Tracts in Natural Philosophy. Springer-Verlag, New York, 1994.

[19] A. Glezer, M. Amitay, Synthetic jets, Annu. Rev. Fluid Mech., 34 (2002) 503–529.

[20] C. Grandmont, B. Maury and A. Soualah, Multiscale modelling of the respiratory tract: A theoretical framework, ESAIM: Proc., 23 (2008), 10–29.
[21] J-W. He, R. Glowinski, R. Metcalfe, A. Nordlander, J. Periaux, Active control and drag optimization for flow past a circular cylinder. 1: oscillatory cylinder rotation, J. Comput. Phys., 163 (2000), 83–117.
[22] J-Y. Hwang, K-S. Yang, S-H. Sun, Reduction of flow-induced forces on a circular cylinder using a detached splitter plate, Phys. Fluids, 15 (2003), 2433–2436.
[23] J. Kim, H. Choi, Distributed forcing of flow over a circular cylinder, Phys. Fluids, 17 (2005), Article ID 033103.
[24] E. Konstantinidis, S. Balabani, M. Yianneskis, The timing of vortex shedding in a cylinder wake imposed by periodic inflow perturbations, J. Fluid Mech. 543 (2005), 45–55.
[25] F. Li, N. Aubry, Feedback control of a flow past a cylinder via transverse motion, Phys. Fluids, 15 (2003), 2163–2176.
[26] Z. Li, I. Navon, M. Hussaini, F. Le Dimet, Optimal control of cylinder wakes via suction and blowing, Comput. Fluids, 32 (2003), 149–171.
[27] J. L. Lions, *"Quelques méthodes de résolution des problèmes aux limites non linéaires"*, Dunod, 2002.
[28] J. Moran, An introduction to theoretical and computational aerodynamics, Dover, 2003.
[29] E.M.D. Ngom, A. Sène and D. Y. Le Roux, Global stabilization of the Navier-Stokes equations around an unstable equilibrium state with a boundary feedback controller, to appear.
[30] E.M.D. Ngom, A. Sène and D. Y. Le Roux, Boundary stabilization of the Navier-Stokes equations with feedback controller via a Galerkin method, Evolution Equations and Control Theory, **3** (2014), 147–166.
[31] S. S. Ravindran, Optimal boundary feedback flow stabilization by model reduction, Comput. Methods Appl. Mech. Engrg. 196 (2007), 2555–2569.
[32] S. S. Ravindran, Stabilisation of Navier-Stokes equations by boundary feedback, Int. J. Numer. Anal. Model. 4 (2007), 608–624.
[33] J.-P. Raymond and L. Thevenet, *Boundary feedback stabilization of the two-dimensional Navier-Stokes equations with finite-dimensional controllers*, Discrete Contin. Dynam. Systems, **27** (2010), 1159–1187.
[34] M. Schäfer, S. Turek , Benchmark Computations of Laminar Flow around a Cylinder, In Flow Simulation with High-Performance Computers II, Notes on Numerical Fluid Mechanics, 1 (1996), 547–566.
[35] A. Sene, B. A. Wane and D. Y. Le Roux, Control of irrigation channels with variable bathymetry and time dependent stabilization rate, C. R. Acad. Sci. Paris Ser. I, **346** (2008), 1119–1122.
[36] R. Temam, Navier-Stokes Equations, Theory and Numerical Analysis, Amer. Math. Soc., Providence, RI, 2001.
[37] P. T. Tokumaru and P. E. Dimotakis, Rotary oscillatory control of a cylinder wake, J. Fluid Mech. 224 (1991), 77–90.